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JONES POLYNOMIAL INVARIANTS

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ABSTRACT

The Jones polynomial is a well-defined invariant of virtual links. We observe the effect of a generalised mutation M of a link on the Jones polynomial. Using this, we describe a method for obtaining invariants of links which are also invariant under M.

The Jones polynomial of welded links is not well-defined in $\mathbb{Z}[q^{1/4}, q^{-1/4}]$. Taking $M = F_o$ allows us to pass to a quotient of $\mathbb{Z}[q^{1/4}, q^{-1/4}]$ in which the Jones polynomial is well-defined. We get the same result for $M = F_u$, so in fact, the Jones polynomial in this ring defines a fused isotopy invariant. We show it is non-trivial and compute it for links with one or two components.

1. Introduction

Virtual links were defined by Kauffman in [5]. There, and independently in [2], it was proved that classical knots embed in virtual knots, so all invariants of virtual knots restrict to classical knots. Kauffman described how to extend the bracket and Jones polynomials to virtual knots in [5].

Let D be a virtual link diagram in $\mathbb{R}^2$, that is a link diagram with an extra type of crossing called virtual crossing. *Virtual isotopy* is equivalence of diagrams under the classical Reidemeister moves R_1, R_2, R_3 and virtual moves V_1, V_2, V_3, V_4 shown in figure 1. However the forbidden moves F_o and F_u which are shown in figure 2 are not allowed.

Welded isotopy is the extension of virtual isotopy which also allows the F_o move. A *welded link* is the equivalence class of a virtual link diagram under welded isotopy. Welded links can be obtained as the closure of welded braids [1], [3].

It is worth noting that the knot group is a welded isotopy invariant and so any knot with non-trivial knot group is not welded isotopic to the unknot. Hence the class of welded knots is not trivial.

Allowing both of the forbidden moves F_o and F_u gives rise to *fused isotopy*, also introduced by Kauffman in [5]. It was shown in [2] and [4] that any virtual knot is fused isotopic to the unknot.

The virtual knot in figure 3 has non-trivial Jones polynomial, therefore it is not *virtually* the unknot. On the other hand, it is the unknot under welded isotopy. This shows that the Jones polynomial, defined in the ring $\mathbb{Z}[q^{1/4}, q^{-1/4}]$ is not an invariant of welded links.

We will find a quotient of this ring where Kauffman's bracket polynomial is a non-trivial regular isotopy invariant and the Jones polynomial is a non-trivial isotopy invariant of welded links. Furthermore we show that this is also invariant under the other forbidden move, F_u, and so it is a fused isotopy invariant.

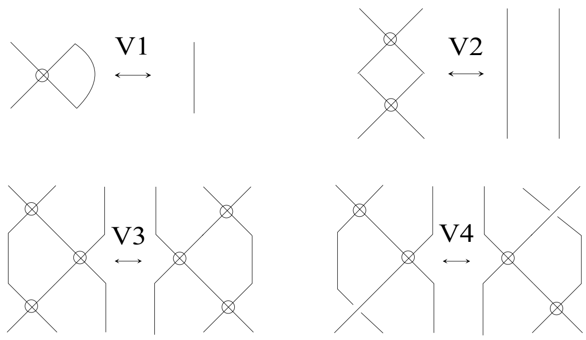

Figure 1: Virtual moves

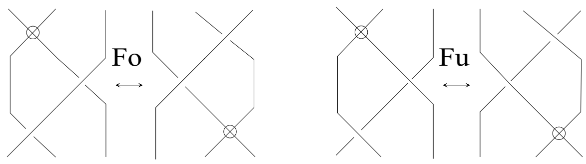

Figure 2: Forbidden moves F_o and F_u

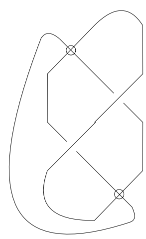

Figure 3: A non-trivial virtual knot which is welded isotopic to the unknot

From now on, a link diagram will mean an unoriented virtual link diagram in $\mathbb{R}^2$ unless stated otherwise. Positive and negative smoothings of a real crossing of a link diagram D are shown in figure 4. Smoothing all of the real crossings of D leaves a virtual link diagram with no real crossings (which may have virtual crossings). Such a diagram is equivalent to the unlink (possibly via some V_1, V_2, V_3 moves). A choice of smoothing of all of the classical crossings is called a *state* of D. Denote the set of all states of D by $\mathcal{S}(D)$. Let $d = -A^2 - A^{-2}$, $\alpha(S)$ = number of positive smoothings, $\beta(S)$ = number of negative smoothings and $t(S)$ = number of circles for each $S \in \mathcal{S}(D)$. Then Kauffman's bracket polynomial is defined to be:

$$\langle S \rangle := d^{t(S)-1} \text{ and}$$

$$\langle D \rangle := \sum_{S \in \mathcal{S}(D)} A^{\alpha(S)-\beta(S)} \langle S \rangle.$$

This is a regular isotopy invariant of virtual links. Orient D and let $w(D)$ be the writhe of the diagram. If L is the link represented by this oriented diagram, then the Kauffman polynomial $p_L(A) := (-A)^{-3w(D)} \langle D \rangle$ is an isotopy invariant. Kauffman showed that this polynomial evaluated at $A = q^{-1/4}$ is the Jones polynomial $V_L(q) \in \mathbb{Z}[q^{1/4}, q^{-1/4}]$.

Figure 4: Positive and negative smoothings

2. General Method

Definition 1. Let D be a link diagram and let B be a disc. Call $T = D \cap B$ a *(generalised) tangle* if D intersects ∂B transversally and $D \cap \partial B$ does not contain any crossing points.

Note that, $D \cap \partial B$ consists of an even number of points which we will call the *end points*. This differs from the usual notion of a tangle since extra circles are allowed, but we will refer to these generalised tangles as tangles throughout.

Definition 2. Let T be a tangle in a diagram D with $2k$ end points on $D \cap \partial B$. Label an arbitrary end point by 1 and label the others $2, 3, \ldots, 2k$ in a clockwise direction. Fix this labelling. Then the *inside permutation* of the end points of the arcs of T can be expressed in a unique way, by $\pi = [1x_2][x_3x_4] \ldots [x_{2k-1}x_{2k}]$, where

1. $x_i \in \{2, \ldots, 2k\}$ for all $i = 2, \ldots, 2k$.
2. $x_{2i-1} < x_{2i+1}$ for $i = 2, \ldots, k-1$.
3. $x_{2i-1} < x_{2i}$ for $i = 2, \ldots, k$.
4. Each $[x_{2i-1}x_{2i}]$ means that the end points enumerated as x_{2i-1}, x_{2i} are joined by an arc of T.

Definition 3. Let D be a link diagram and let T denote a tangle in D. Write $D = T \cup (D - T)$. Then a *state of* T is a choice of smoothing of all of the classical crossings in T, and a *state of* $D - T$ is a choice of smoothing of all of the classical crossings in $D - T$.

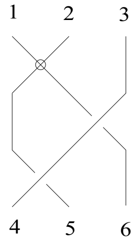

Figure 5: The inside permutation of a tangle

Since T may contain real crossings, a state $S \in \mathcal{S}(D-T)$ is a possibly non-trivial link which we denote by $T(S)$. $T(S)$ contains T as a tangle and has no real crossings in the complement of T.

Definition 4. Let T be a tangle in a diagram D. The *outside permutation* of T is the permutation of the endpoints of T induced be the arcs outside B.

Definition 5. Let D be a link diagram which contains a tangle T in a disc B. Remove T and replace it with another tangle T' where $T \cap \partial B = T' \cap \partial B$. This operation is called a *generalised mutation* and we will denote this move in a diagram by M. Note that the replacement of T' by T is also considered an M move, as usual.

Theorem 1. Suppose D' is obtained from D by an M move which replaces the tangle T by T'. Then

$$\langle D\rangle - \langle D'\rangle = \sum_{S\in\mathcal{S}(D-T)} (\langle T(S)\rangle - \langle T'(S)\rangle)\, h_S(A) \qquad \text{where } h_S(A)\in \mathbb{Z}[A, A^{-1}].$$

Proof. $\langle D\rangle = \sum_{S\in\mathcal{S}(D)} A^{\alpha(S)-\beta(S)}\langle S\rangle = \sum_{S\in\mathcal{S}(D-T)} A^{\alpha(S)-\beta(S)}\langle T(S)\rangle$.
Similarly $\langle D'\rangle = \sum_{S\in\mathcal{S}(D'-T')} A^{\alpha(S)-\beta(S)}\langle T'(S)\rangle$. Since $D-T = D'-T'$, they have the same states and the result follows. □

Let T and T' be two tangles which agree on the boundary of the disc B. Consider all of the possible ways of joining the end points of ∂B outside B without introducing any classical crossings. Each of these joinings give link diagrams $D_T,\ D_{T'}$ which are identical outside B. In other words, $D_{T'}$ is obtained by a mutation M from the diagram D_T where D_T is a link diagram obtained by joining the end points of T. Compute $\langle D_T\rangle - \langle D_{T'}\rangle$ for each possibility. Let I_M be the ideal of $\mathbb{Z}[A, A^{-1}]$ generated by these polynomials.

Theorem 2. Let D and $\hat{D}$ be diagrams which are equivalent under regular isotopy extended by the M move which replaces tangle T by T'. Then $\langle D\rangle - \langle\hat{D}\rangle \in I_M$.

Proof. Suppose $\hat{D}$ is obtained from D by a finite sequence, $D = D_0 \to D_1 \to \ldots \to D_N = \hat{D}$, of $R_1, R_2, V_1, V_2, V_3, V_4$ and M moves. Suppose also that $D_{k-1} \xrightarrow{f} D_k$. If

1. $f \in \{R_2, R_3, V_1, V_2, V_3, V_4\}$, then the two diagrams are regular isotopic and we have $\langle D_k\rangle = \langle D_{k-1}\rangle$.
2. $f = M$ then, by theorem 1, $\langle D_k\rangle - \langle D_{k-1}\rangle \in I_M$. □

Corollary 1. The bracket polynomial is a regular isotopy invariant, which is also invariant under the M move in the ring $\mathbb{Z}[A, A^{-1}]/I_M$. □

Define $I_M(q^{-1/4}) := \{p(q^{-1/4}) \in \mathbb{Z}[q^{1/4}, q^{-1/4}] \mid p(A) \in I_M\}$.

Corollary 2. Suppose M is a generalised mutation which leaves the writhe of the diagram invariant. Then the Jones polynomial of an oriented link in the quotient ring $\mathbb{Z}[q^{1/4}, q^{-1/4}]/I_M(q^{-1/4})$ is an isotopy invariant which is also invariant under the M move. □

3. Application to welded and fused links

We apply the theory, using the forbidden move F_o as the M move. An F_o move replaces the tangle T with P, which are shown in figure 6. We will find the ideal I_o generated by $\langle D_T\rangle - \langle D_P\rangle$.

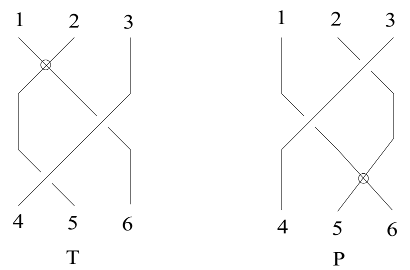

Figure 6: Tangles T and P

From now on T and P will denote these specific tangles, lying in the disc B and whose end points are labelled as in the figure. Let us compute $\langle D_T\rangle - \langle D_P\rangle$ for all possibilities. The states of T and P (within B) are shown in figures 7 and 8 respectively.

To compute $\langle D_T\rangle - \langle D_P\rangle$ the other information required is the number of circles in each state. Let us denote the four possible states of T and P by T_i and P_i, for $i = 1, 2, 3, 4$, respectively. Outside B the end points of any T_i or P_i are joined by arcs which may contain virtual crossings.

Since the tangles T and P have 3 arcs each, then any T_i or P_i can have either 1,2 or 3 components. We can work out exactly how many by looking at the information about the joining of end points as follows. Suppose that inside B we have the permutation $[1a][bc][de]$. Then in order to obtain 3 components we must have exactly the same outside permutation. To obtain 2 components, we must have exactly one of the transpositions $[1a]$, $[bc]$ or $[de]$ in the outside permutation. Otherwise we have only 1 component.

Outside B, there are 15 ways of joining the ends of the 3 arcs of any T_i or P_i. We need to find the number of circles t_i, p_i in T_i, P_i in each case. Looking at figure 7, we see that inside B

$$T_1 \leftrightarrow [15][26][34],\ T_2 \leftrightarrow [13][26][45],\ T_3 \leftrightarrow [12][34][56],\ T_4 \leftrightarrow [13][24][56].$$

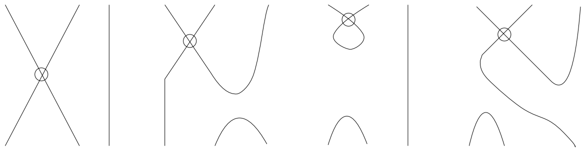

Figure 7: States of T

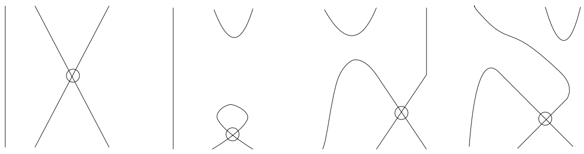

Figure 8: States of P

Looking at figure 8, we see that inside B

$$P_1 \leftrightarrow [16][24][35],\ P_2 \leftrightarrow [16][23][45],\ P_3 \leftrightarrow [12][35][46],\ P_4 \leftrightarrow [15][23][46].$$

We also have the formula $\langle D_T\rangle - \langle D_P\rangle = A^2(d^{t_1-1} - d^{p_1-1}) + (d^{t_2-1} - d^{p_2-1}) + (d^{t_3-1} - d^{p_3-1}) + A^{-2}(d^{t_4-1} - d^{p_4-1})$. In the table below, we find the number of circles outside B for each case and compute $\langle D_T\rangle - \langle D_P\rangle$.

	t_1, p_1	t_2, p_2	t_3, p_3	t_4, p_4	$\langle D_T\rangle - \langle D_P\rangle$
[12][34][56]	2,1	1,1	3,2	2,1	$g_1(A)$
[12][35][46]	1,2	1,1	2,3	1,2	$-g_1(A)$
[12][36][45]	1,1	2,2	2,2	1,1	0
[13][24][56]	1,2	2,1	2,1	3,1	$g_2(A)$
[13][25][46]	1,1	2,1	1,2	2,2	0
[13][26][45]	2,1	3,2	1,1	2,1	$g_1(A)$
[14][23][56]	1,1	1,2	2,1	2,2	0
[14][25][36]	1,1	1,1	1,1	1,1	0
[14][26][35]	2,2	2,1	1,2	1,1	0
[15][23][46]	2,1	1,2	1,2	1,3	$-g_2(A)$
[15][24][36]	2,2	1,1	1,1	2,2	0
[15][26][34]	3,1	2,1	2,1	1,2	$g_2(A^{-1})$
[16][23][45]	1,2	2,3	1,1	1,2	$-g_1(A)$
[16][24][35]	1,3	1,2	1,2	2,1	$-g_2(A^{-1})$
[16][25][34]	2,2	1,2	2,1	1,1	0

A simple calculation shows that $g_1(A) = 0$ and $g_2(A) = A^{-6} - A^{-2} - 1 + A^4$. Also notice that $g_2(A^{-1}) = A^6 - A^2 - 1 + A^{-4} = A^2 g_2(A)$.

Corollary 3. If D and D' are regular isotopic welded diagrams, then $\langle D\rangle \equiv \langle D'\rangle$ in the quotient ring $R := \mathbb{Z}[A, A^{-1}]/I_o$ where I_o is the ideal generated by $g(A) := A^{-4} - 1 - A^2 + A^6$. Hence the bracket polynomial is a regular welded isotopy invariant in the ring R.

Proof. Put $M = F_o$. Then $\langle D_T\rangle - \langle D_P\rangle \in \{0, \pm g(A), \pm A^{-2}g(A)\}$ follows from the above calculations. Now apply theorem 2. □

Corollary 4. The Jones polynomial of an oriented welded link in the quotient ring $\mathbb{Z}[q^{1/4}, q^{-1/4}]/I_o(q^{-1/4})$ is a welded isotopy invariant (we also denote this ring by R).

Proof. An F_o move does not change the writhe of the diagram. □

Welded isotopy allows the F_o move, but not the F_u move. Suppose we define an equivalence of diagrams allowing the F_u move instead of the F_o move. Consider the tangles T and P in figure 6. By changing all the over crossings to undercrossings in both tangles we get two new tangles T' and P'. Notice that the states of T' are the same as states of T with coefficient A replaced by A^{-1}. This gives us $\langle D_{T'}\rangle = A^2 d^{t_4} + d^{t_3} + d^{t_2} + A^{-2} d^{t_1}$. Similarly for P'. So to compute $\langle D_{T'}\rangle - \langle D_{P'}\rangle$, we replace A by A^{-1} in $\langle D_T\rangle - \langle D_P\rangle$. Therefore we obtain:

Corollary 5. The Jones polynomial of an oriented link is invariant under isotopy extended by the F_u move in the ring R.

Proof. By the above observations $\langle D_{T'}\rangle - \langle D_{P'}\rangle \in \{0, \pm g(A^{\pm 1})\}$. But $g(A^{-1}) = A^2 g(A)$ so $\langle D_{T'}\rangle - \langle D_{P'}\rangle$ generates the same ideal as $\langle D_T\rangle - \langle D_P\rangle$. □

Hence if we allow both of the moves F_o and F_u, we get:

Theorem 3. The Jones polynomial of fused links in R is a fused isotopy invariant.

Proof. $I_o = I_u$ where I_u is the ideal from the F_u move. □

According to the result of Kanenobu [4] and Goussarov-Polyak-Viro [2], any virtual knot is fused isotopic to the unknot. As a result, we have:

Theorem 4. The Jones polynomial of any virtual knot in R is 1. □

Corollary 6. Let D be any knot diagram. Then $\langle D\rangle \equiv (-A)^{3w(D)} \bmod I_o$.

Proof. Since the Kauffman polynomial of any virtual knot is 1 mod I_o, we have $(-A)^{-3w(D)}\langle D\rangle \equiv 1 \bmod I_o$. □

Definition 6. Let D be an oriented virtual link diagram with components l_i. We define the *linking number* $lk(l_i, l_j) := \frac{1}{2}$(algebraic sum of the classical crossings of components l_i and l_j). If D has n components, we call the sum of all $lk(l_i, l_j),\ 1 \le i < j \le n$ the *total linking number*.

This definition agrees with the linking number of classical links and is invariant under V_k for $k = 2, 3, 4$, F_o and F_u moves. Therefore the (total) linking number is well defined for virtual links, welded links and fused links. Of course, this number need not be an integer anymore. For example the linking number of the components of the virtual right Hopf link VRH, shown in figure 9b, is 1/2.

Theorem 5. Let L be an oriented virtual link with 2 components l_1, l_2 such that $lk(l_1, l_2) = k \in \frac{1}{2}\mathbb{Z}$. Then the Jones polynomial of L is $V_L(q) \equiv -q^{1/2} - q^{3k-1/2} \bmod I_o$.

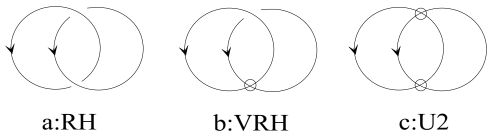

Figure 9: Three distinct fused links

Proof. Let D be an oriented diagram of L and let D_i correspond to the components l_i of L, for $i = 1, 2$. Then the writhe of D is $w(D) = w(D_1) + w(D_2) + 2k$ where $w(D_i)$ denotes the algebraic sum of the crossings of D_i with itself. Consider a real crossing between D_1 and D_2 (if there are no real crossings, simply introduce one by an R_2 move). Without loss of generality, suppose this crossing is positive. Smoothing D at this crossing, we get $\langle D\rangle = A\langle K_1\rangle + A^{-1}\langle K_2\rangle$. Then the Kauffman polynomial of L is $p_L(A) = (-A)^{-3w(D)}(A\langle K_1\rangle + A^{-1}\langle K_2\rangle)$ where K_1 and K_2 are both knot diagrams. Therefore by corollary 6, we obtain $p_L(A) \equiv (-A)^{-3w(D)}(A(-A)^{3w(K_1)} + A^{-1}(-A)^{3w(K_2)})$ mod I_o. We obtain K_1 by positive smoothing, hence it induces the orientation of D and it has one less positive crossing, so $w(K_1) = w(D) - 1$. K_2 does not induce an orientation from D. Orient K_2. Recall that the writhe of a knot is independent of its orientation. The orientation on K_2 will agree with the orientation of one of D_1 or D_2 and disagree with the other one. Therefore the linking number will make a negative contribution to $w(K_2)$. So we can compute $w(K_2) = w(D_1) + w(D_2) - (2k - 1) = w(D) - 4k + 1$. Substituting back, $p_L(A) \equiv -A^{-2} - A^{-12k+2}$ mod I_o. Then the Jones polynomial of L is $V_L(q) \equiv -q^{1/2} - q^{3k-1/2}$ mod I_o. □

4. Computations

Given two virtual links L and L', we develop a method to test if $p_L \equiv p_{L'}$ mod I_o.

Definition 7. Represent a polynomial $\Sigma_{i=-n}^{m} c_i A^i \in \mathbb{Z}[A, A^{-1}]$ by its *coefficient sequence*, which is the bi-infinite sequence with finitely many nonzero terms and whose k-th term corresponds to the coefficient of A^k.

Definition 8. Let r_m denote the sequence $(\ldots, 0, 1, 0, 0, 0, -1, 0, -1, 0, 0, 0, 1, 0, \ldots)$, where the last nonzero term is the coefficient of A^m. In particular, r_6 corresponds to $g(A)$.

Lemma 1. For each m, $r_m = (\ldots, 0, 1, 0, 0, 0, -1, 0, -1, 0, 0, 0, 1, 0, \ldots) \in I_o$.

Proof. The ideal I_o is generated by $g(A) = A^{-4} - 1 - A^2 + A^6$. Thus any polynomial of the form $A^k g(A) = A^{k-4} - A^k - A^{k+2} + A^{k+6} \in I_o$, for any $k \in \mathbb{Z}$. □

Theorem 6. Let L and L' be virtual links and let $u,\ v$ be the coefficient sequences of p_L and $p_{L'}$ respectively. Then $p_L \equiv p_{L'}$ mod I_o if and only if $u - v = \Sigma_{i=1}^{k} c_i r_{n_i}$ where $n_i \in \mathbb{Z}$.

Proof. Suppose $p_L \equiv p_{L'}$ mod I_o. Then $p_L - p_{L'} = \Sigma_{i=-n}^{m} c_i A^i g(A)$. Therefore $u - v = \Sigma_{i=-n}^{m} c_i r_{6+i}$.

Conversely, if $u - v = \Sigma_{i=1}^{k} c_i r_{n_i}$, then the right hand side of the equation is clearly in I_o, and so the left hand side must also be in I_o. □

Definition 9. Two coefficient sequences u, v are called *equivalent* in R if they represent equivalent polynomials in R.

Lemma 2. Every equivalence class of coefficient sequences has a representative of the form $(\ldots, 0, c_{-4}, \ldots, c_6, 0, \ldots)$.

Proof. Suppose we have a polynomial p in R and a representative of it is given by the coefficient sequence $u_m = (\ldots, 0, c_n, \ldots, c_m, 0, \ldots)$. If $m > 6$, then compute $u_k = u_m - c_m r_m$. Clearly u_m and u_k are equivalent and $k < m$. Note that c_l for $l < m - 10$ are unchanged by this operation. In particular, c_l for $l < -4$ are not changed. Applying this operation finitely many times gives a coefficient sequence which is equivalent to u_m and whose last nonzero term has index ≤ 6. Similarly, we can eliminate terms c_l, for $l < -4$ without effecting terms with index $k > 6$. This sequence corresponds to a representative of $\overline{p}$, whose highest degree is at most A^6 and lowest degree at least A^{-4}. □

For example, $g(A)$ is such a representative. But a representative written in this form is not unique.

Lemma 3. Every equivalence class of coefficient sequences has a unique representative of the form $(\ldots, 0, c_{-4}, \ldots, c_6, 0, \ldots)$, with $c_0 = 1$. That is, every polynomial in R can be written uniquely in the form $c_{-4}A^{-4} + \ldots + c_6 A^6$ with $c_0 = 1$.

Proof. Given a coefficient sequence, lemma 2 shows there is a representative of it in the form $u = (\ldots, 0, c_{-4}, \ldots, c_6, 0, \ldots)$. Then $u + (c_0 - 1)r_6$ is a representative in the required form.

To show uniqueness, we first note that the coefficient sequence $-r_6$ satisfies the conditions of the lemma and corresponds to the generator g of the ideal I_o. Suppose that there is another sequence u satisfying these conditions and $u + r_6 = \Sigma_{i=1}^{k} c_i r_{n_i}$. If $k = 1$ then since u satisfies the conditions, $n_1 \leq 6$ and $n_1 - 10 \geq -4$ which lead to $n_1 = 6$. So $u = (c_1 - 1)r_6$. But we must have $c_1 - 1 = -1$, hence $u = -r_6$. Now suppose $k > 1$. Without loss of generality, assume that $c_i \neq 0$ for all i and $n_1 < \ldots < n_k$. Then we have $-4 \leq n_1 - 10 < n_1 < n_k \leq 6$ leading to a contradiction. Hence $-r_6$ is the unique representative of g satisfying the conditions. Therefore each polynomial class in R has a unique representative in the desired form. □

Below, we compute the Kauffman polynomial of some virtual links, and find their unique representatives modulo I_o. Let us denote the unique form of a polynomial p_L by $\overline{p}_L$.

Example 1. Using theorem 5, we compute the polynomials of the links shown in figure 9. The linking number of the components of the right hopf link RH is 1. Now $p_{RH}(A) \equiv -A^{-10} - A^{-2} \bmod I_o$. We find its standard form in R as follows: Ignoring the zeros at the beginning and at the end

	k		-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4
	u_{-2}	=	-1	0	0	0	0	0	0	0	-1						
+	r_0	=	1	0	0	0	-1	0	-1	0	0	0	1				
	u_0	=	0	0	0	0	-1	0	-1	0	-1	0	1				
+	r_4	=					1	0	0	0	-1	0	-1	0	0	0	1
	u_4	=					0	0	-1	0	-2	0	0	0	0	0	1

We add $-r_6$ to u_4 to get the unique representative which yields $\overline{p}_{RH}(A) = -2A^{-4} - 2A^{-2} + 1 + A^2 + A^4 - A^6$.

The linking number of the components of VRH is 1/2. So $p_{VRH}(A) \equiv -A^{-4} - A^{-2}$. We put it in its standard form by adding $-r_6$ to its coefficient sequence and we get $\bar{p}_{VRH}(A) = -2A^{-4} - A^{-2} + 1 + A^2 - A^6$. $U2$ is the unlink with two components, so $p_{U2}(A) = -A^{-2} - A^2$ and its standard form is $\overline{p}_{U2}(A) = -A^{-4} - A^{-2} + 1 - A^6$.

These computations show that the polynomials of the three links in figure 9 are mutually different. We conclude that the Jones polynomial mod I_o of welded (or fused) links is a non-trivial isotopy invariant.

Example 2. Let L_1 and L_2 be the two links shown in figure 10. Smoothing a positive crossing between the first and second components and then applying theorem 5, we can easily compute $p_{L_1}(A) \equiv A^{-12} + A^{-4} + 1 + A^{16}$ and $p_{L_2}(A) \equiv A^{-6} + A^{-4} + 1 + A^{10}$. These are indeed different polynomials since $(\overline{p_{L_1} - p_{L_2}})(A) = -A^{-2} + 1 + A^4 - A^6$. This example shows that the Kauffman polynomial is not a function of the total linking number.

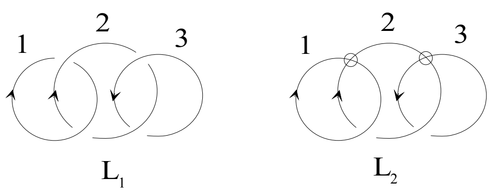

Figure 10: Two links with the same total linking number

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