Classifying links under fused isotopy

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ABSTRACT
All knots have been shown to be isotopic to the unknot using a process known as virtualization. We extend and adapt this process to show that, up to fused isotopy, classical links are classified by their linking numbers. We provide an algebraic proof, utilising Alexander’s Theorem and some simple results about the pure braid group.

Keywords: Virtual and fused braids, virtual and fused links, linking number, fused isotopy

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1. Introduction
Classical braids and links have been generalized to the virtual category [10] – adding virtual crossings and extending isotopy to allow the virtual analogues of the classical Reidemeister moves. The forbidden moves $F_0$ and $F_u$, shown in Fig. 1, are not allowable under virtual isotopy. Extending virtual isotopy in the virtual braid group $\mathcal{V}B_n$ to allow the $F_0$ move gives rise to the welded braid group $\mathcal{W}B_n$, which has been shown to be isomorphic to $\mathcal{P}C_n$, the group of automorphisms of the free group on $n$ elements of permutation-conjugacy type [4]. Allowing both of the forbidden moves $F_0$ and $F_u$ gives rise to fused isotopy [10]. That is, two virtual links $L_1$ and $L_2$ are called fused isotopic if $L_2$ can be obtained from $L_1$ by a finite sequence of Reidemeister moves, virtual moves and $F_0$, $F_u$ moves.

![Fig. 1. The forbidden moves](image-url)

Let $\mathcal{K}$ be the space of classical links embedded in $S^3$ and let $\mathcal{V}\mathcal{K}$ be the space of virtual links. Kauffman [10], and independently Goussarov-Polyak-Viro [7], have shown that $\mathcal{K}$ embeds into $\mathcal{V}\mathcal{K}$. Let $f$ denote the natural inclusion of $\mathcal{V}\mathcal{K}$ into the space of fused links
If \( K \). Then we have \( K \xrightarrow{f} \mathcal{V} \xrightarrow{f} K \), and when we refer to a classical link (under fused isotopy) we mean \( f \circ i(\mathcal{L}) \), the image of a link \( \mathcal{L} \in K \) in the space \( \mathcal{F} K \).

In [9], Kanenobu showed that all knots are fused isotopic to the unknot. He showed that all of the classical crossings of a virtual knot can be virtualized; that is every classical crossing can be changed into a virtual crossing by applying a sequence of fused isotopy moves. However, crossings between different components of a link cannot be virtualized using the same methods. The following theorem from [9] provides us with allowable moves under fused isotopy which were used in the virtualization procedure.

**Theorem 1.** The moves \( M_1 \), \( M_2 \) and \( M_3 \), shown in Fig. 2, can be realised by fused isotopy.

![Fig. 2. Allowable moves in fused isotopy](image)

In [5] the authors showed that the Jones polynomial for welded and fused links is well-defined in a quotient of \( Z[A, A^{-1}] \) and observed that this polynomial depends only upon the linking number for links with two components. Inspired by this, we show that classical links, under fused isotopy, can be determined by the linking number of their components.

**Theorem 2.** A classical link \( \mathcal{L} \) with \( n \)-components is completely determined by the linking numbers of each pair of components under fused isotopy.

The strategy that we use to prove Theorem 2 is to write \( \mathcal{L} \) as the closure of a braid \( \alpha \) on \( m \) strands (where \( m \geq n \)) and then to transform \( \alpha \) into a pure braid \( \beta \) on \( n \) strands whose closure is also \( \mathcal{L} \). We show that \( \beta \) depends only on the linking numbers of the components of \( \mathcal{L} \). This means that any classical link with the same linking numbers as \( \mathcal{L} \) can be obtained as the closure of \( \beta \). We need some preliminaries before we proceed with the proof.

### 2. Preliminaries

Recall that an element of the pure braid group \( P_n \) is an \( n \)-strand braid where the permutation induced by the strings is the identity. The pure braid group \( P_n \) has a presentation with generators \( A_{i,j} \) with \( 1 \leq i < j \leq n \) where

\[
A_{i,j} = \sigma_{j-1} \sigma_{j-2} \cdots \sigma_{j-i+1}^{-1} \sigma_{j-1}^{-1} \cdots \sigma_{j-i+2}^{-1} = \sigma_{i-1} \sigma_{i+1}^{-1} \cdots \sigma_{j-i+1}^{-1} \cdots \sigma_{j-i} \sigma_i.
\]
In particular, we have \( A_{j,j+1} = \sigma_j^2 \). Let \( U_k \) be the subgroup of \( P_n \) generated by \( \{A_{i,k} : 1 \leq i < k\} \). Then every element of \( P_n \) can be written in the unique normal form \( x_2 x_3 \ldots x_n \), where \( x_k \in U_k \) (see [2] for details). Define \( B_{i,j} := \sigma_{j-1} \ldots \sigma_{i+1} \sigma_i \) for \( i < j \) and \( B_{i,j} := 1 \). Then by definition \( A_{i,j+1} = B_{i,j}^{-1} A_{j,j+1} B_{i,j} \), and we can see from Fig. 3 that for \( k < i < j \), \( B_{i,j} \) commutes with \( A_{k,i+1} \) in \( B_n \).

![Fig. 3. \( A_{k,j+1} B_{i,j} = B_{i,j} A_{k,j+1} \)]

The virtual braid group on \( n \)-strands, \( V B_n \), can be defined by adding extra generators \( t_i \), for \( 1 \leq i \leq n \), corresponding to the virtual crossings, and relations corresponding to the virtual isotopy moves \([8, 10] \).

We define the **fused braid group**, \( F B_n \), by adding the relations \( \sigma_i^{-1} t_j \sigma_i = \sigma_i t_i \sigma_j^{-1} \) with \( |i - j| = 1 \), to the virtual braid group \( V B_n \). If \( j = i + 1 \) the relation corresponds to the \( F_0 \) move, and if \( i = j + 1 \) it corresponds to the \( F_0 \) move. The **fused pure braid group**, \( FP_n \), is the group of fused braids for which the induced permutation is the identity.

The explicit realization of the moves \( M_1 \), \( M_2 \) and \( M_3 \) using \( F_0 \) and \( F_0 \) moves is shown in [9], and this gives rise to the following consequences in \( F B_n \):

\[
\begin{align*}
M_1 : & \quad \sigma_i t_j \sigma_i = \sigma_i t_i \sigma_j^{-1} \\
M_2 : & \quad \sigma_i^{-1} t_j \sigma_j^{-1} = \sigma_j^{-1} t_i \sigma_i^{-1} \\
M_3 : & \quad \sigma_i \sigma_j^{-1} \sigma_i = \sigma_j \sigma_i^{-1} \sigma_j,
\end{align*}
\]

where \( |i - j| = 1 \).

The following lemmas are used in the proof of Theorem 2, and the indices have been chosen to match the usage in the proof. Let \( \sim \) denote the equivalence class generated by fused isotopy and let \( \mathcal{B}_k = U_k \simeq \sim \). In the following, we refer to application of the braid relations \( \sigma_i \sigma_j^\delta = \sigma_j \sigma_i^\delta \) for \( |i - j| > 1 \), and \( \theta, \delta \in \{1, -1\} \), as **commutations in** \( B_n \), and the use of the virtual braid relations \( \sigma_i^\tau_j^\delta = t_i^\delta \sigma_j^\tau \) for \( |i - j| > 1 \), and \( \delta, \epsilon \in \{1, -1\} \), as **commutations in** \( V B_n \).

**Lemma 0.1.** In the fused pure braid group \( FP_n \) we have:

\[
A_{j,j+1} A_{i,j+1}^{-1} A_{i,j+1}^{-1} = A_{i,j+1} \quad \text{where} \quad 1 \leq i < j + 1 \leq n.
\]

(2.2)

In other words, \( A_{j,j+1} = \sigma_j^2 \) is in the centre of \( \mathcal{B}_j + 1 \).

**Proof.** Using the relations \( \sigma_j \sigma_{j-1}^{-1} \sigma_j = \sigma_j^{-1} \sigma_{j-1}^{-1} \sigma_{j-1} \) corresponding to an \( M_3 \) move, and \( \sigma_j \sigma_{j-1}^{-1} \sigma_j^{-1} = \sigma_{j-1}^{-1} \sigma_j \sigma_{j-1}^{-1} \) corresponding to an \( R_3 \) move, we obtain
Using Eq. (2.3), we can compute

\[ A_{i,j+1}A_{i,j+1}^{-1} = A_{i,j+1}B^{-1}_{i,j}A_{i,j+1}^{-1} \]

\[ = \sigma_j^2 \sigma_j^{-1} \sigma_j^2 \sigma_{j-1} \sigma_j^{-2} \]

\[ = \sigma_j \sigma_j^{-1} \sigma_j^{-1} \sigma_j \sigma_j^{-1} \sigma_j^{-1} \sigma_j \]

\[ = \sigma_j \sigma_j^2 \sigma_j \]

\[ \tag{2.3} \]

Lemma 0.2. For every \( 1 \leq j + 1 < n \), the subgroup \( B_{j+1} \) of \( F_P \) is commutative.

Proof. Assume without loss of generality that \( k < i \). Then

\[ A_{k,j+1}A_{i,j+1}^{-1} = A_{k,j+1}B^{-1}_{i,j+1}A_{i,j+1}B_{i,j} \]

\[ = B_{i,j}A_{k,j+1}A_{i,j+1}B_{i,j} \text{ by commutation in } B_n \]

\[ = B_{i,j}A_{k,j+1}A_{k,j+1}B_{i,j} \text{ by Lemma 0.1} \]

\[ = B_{i,j}A_{i,j+1}A_{k,j+1} \text{ by commutation in } B_n \]

\[ = A_{i,j+1}A_{k,j+1}. \]

Lemma 0.3. In \( F_B \), we have:

\[ A_{i,j+1}t_j = t_j A_{i,j} \text{ for } 1 \leq i \leq j - 1. \]  \( \tag{2.4} \)

Proof. Using the relations \( \sigma_j \sigma_{j-1} t_j = t_{j-1} \sigma_j \sigma_{j-1} \) corresponding to an \( F_0 \) move, and \( \sigma_{j-1} \sigma_j t_{j-1} = t_j \sigma_{j-1} \sigma_j^{-1} \) corresponding to an \( M_1 \) move, we obtain

\[ A_{i,j+1}t_j = B_{i,j+1}^{-1} \sigma_j \sigma_{j-1} \sigma_j \sigma_{j-1} \]

\[ = B_{i,j+1}^{-1} \sigma_j \sigma_{j-1} \sigma_j \sigma_{j-1} t_j \]

\[ = B_{i,j+1}^{-1} \sigma_j \sigma_{j-1} \sigma_j \sigma_{j-1} t_j B_{i,j-1} \text{ by commutation in } V B_n \]

\[ = B_{i,j+1}^{-1} \sigma_j \sigma_{j-1} \sigma_j \sigma_{j-1} t_j B_{i,j-1} \text{ by an } F_0 \text{ move} \]

\[ = B_{i,j+1}^{-1} \sigma_j \sigma_{j-1} \sigma_j \sigma_{j-1} t_j B_{i,j-1} \text{ by an } M_1 \text{ move} \]

\[ = t_j B_{i,j+1}^{-1} \sigma_j \sigma_{j-1} B_{i,j-1} \text{ by commutation in } V B_n \]

\[ = t_j A_{i,j}. \]
3. Proof of Theorem 2

Let L be a classical link with n-components. By Alexander’s Theorem, there exists α ∈ B_m with m ≥ n, such that the closure of α is L. Chow [3] (or see page 22 of [2]) shows that every α can be written in the form α = x_1B_{k_1,2} ... x_mB_{k_m,m} where x_i ∈ U_i ≤ P_m and 1 ≤ k_i ≤ 1. Let ̂α denote the closure of α. If m > n then we will construct ̂β ∈ B_n such that ̂β is fused isotopic to ̂α.

If B_{k_i,i} = 1 for all i = 2, ... , m then α is a pure braid, which means that m must be equal to n. So let us assume that B_{k_s,s} = 1, for some s, and that if i > s then B_{k_i,i} = 1. This means that the permutation induced by α is the identity on the strands s + 1, ... , m. Therefore, each of these strands forms a separate component of the link ̂α. Now, conjugating α with B_{1,m} gives B_{1,m}^−1 αB_{1,m}, and as shown in Fig. 4, the (m − 1)-st strand of the original braid α becomes the m-th strand of the new braid.

Thus if we conjugate α with B_{1,m} (m − s) times, we get α' = B_{1,m}^{t=m} αB_{1,m}^{s=m} and the s-th strand of α becomes the m-th strand of α'. Since α' is just a conjugate of α their closures are isotopic. Now write α' = y_2B_{t_2,2} ... y_mB_{t_m,m} with y_i ∈ U_i. Then B_{t_m,m} = 1 and so by definition, B_{t_m,m} = σ_{m−1}B_{t_m,m−1}. A picture of α' is shown in Fig. 5, where W = y_2B_{t_2,2} ... y_mB_{t_m,m−1}.

Fig. 4. B_{1,m}^{−1} αB_{1,m}

Fig. 5. The braid α'
Since $U_m$ is commutative (by Lemma 0.2), we can write
\[ y_m = A_{1,m}^{r_1} \cdots A_{m-2,m}^{r_{m-2}} A_{m-1,m}^{r_{m-1}} \] for some $r_1, \ldots, r_{m-1}$.

By definition, $A_{m-1,m}^{r_{m-1}} = \sigma_m^{2r_{m-1}}$, and since $B_{t_{m-1, m}} = \sigma_m B_{t_{m-1, m}}$, we obtain
\[ y_m B_{t_{m-1, m}} = A_{1,m}^{r_1} \cdots A_{m-2,m}^{r_{m-2}} \sigma_m^{2r_{m-1} + 1} B_{t_{m-1, m}}. \]

Since $W$ does not involve the $m$-th strand and $y_m$ is a pure braid, Fig. 5 shows that the $m$-th strand and the other strand that is involved in the last occurrence (and hence in all of the previous occurrences) of $\sigma_{m-1}$ in $\alpha'$ belong to the same component of $L = \tilde{\alpha}'$. Therefore, following the strategy in [9], we can virtualize all of the $2r_{m-1} + 1$ crossings in $\tilde{\alpha}'$ which correspond to $\sigma_m^{2r_{m-1} + 1}$ in $\alpha'$. In doing so we have not changed the fused isotopy class of $L$ but we have obtained $\tilde{L}$ as the closure of
\[ \alpha_1 = W A_{1,m}^{r_1} \cdots A_{m-2,m}^{r_{m-2}} \sigma_m^{2r_{m-1} + 1} B_{t_{m-1, m}}. \]

By Lemma 0.3, we obtain
\[ \alpha_1 = W \tau_{m-1} V_{m-1} B_{t_{m-1, m}} \]
where $V_{m-1} = A_{1,m-1}^{r_1} \cdots A_{m-2,m-1}^{r_{m-2}}$ which is an element of $U_{m-1}$.

Figure 6 shows that there is only one crossing involving the $m$-th strand in the braid $\alpha_1$. This is the occurrence of $\tau_{m-1}$. In $\tilde{\alpha}_1$, we can get rid of the virtual crossing corresponding to $\tau_{m-1}$ with a virtual move (of type I). We have obtained a new link diagram $\tilde{\alpha}_2$ where
\[ \alpha_2 = W V_{m-1} B_{t_{m-1, m}} \]
has $m-1$ strands and $\tilde{\alpha}_2$ is fused isotopic to $L$.

![Diagram](image_url)

Fig. 6. The braid $\alpha_1$

If we continue with this process, eventually we will get a braid $\beta$ in $B_n$ whose closure is fused isotopic to $L$. Note that since $\beta$ has $n$ strands and $L$ has $n$ components, each strand of $\beta$ corresponds to a different component of $L$ and therefore $\beta$ must be a pure braid. For
i < j, define the group homomorphism $\delta_{i,j} : PB_n \rightarrow \mathbb{Z}$ by

$$\delta_{i,j}(A_{n,t}) = \begin{cases} 1 & \text{if } s = i \text{ and } t = j \\ 0 & \text{otherwise.} \end{cases}$$

Since $\beta$ is a pure braid it is easy to see that $\delta_{i,j}(\beta) = \text{lk}(\xi_i, \xi_j)$ where $\xi_i$ and $\xi_j$ are the corresponding components of $\beta$.

This proves that any classical link $L$ with $n$-components can be obtained as the closure of a pure braid $\beta = x_2 \ldots x_n$ and since $B_k$ is commutative for every $k$, each $x_k$ can be written in the form $x_k = A_{1,k}^{\delta_{1,k}} \ldots A_{k-1,k}^{\delta_{k-1,k}}$ where $\delta_{i,k}$ denotes $\delta_{i,k}(\beta)$. This shows that $\beta$ depends only on the linking number of the components.

4. Discussion

Theorem 2 does not immediately generalize to non-classical links where there are virtual crossings between different components. For example, let $U_2$ be the trivial link with two components and let $L = a$ where $a = \sigma_1 \tau_1 \sigma_1^{-1} \tau_1^{-1} \in FB_2$. Both of these links have (classical) linking number zero, but they are not fused isotopic. In [1] they point out that they can be distinguished by the refinement of the classical linking number via the pair of (virtual) linking numbers of [7]. This is given by ordering the two components and then: (i) counting the sum of the signs of the classical crossing for which component 1 goes over component 2, (ii) counting the sum of the signs of the classical crossing for which component 2 goes over component 1; the classical linking number is then recoverable from this pair.

The work in this paper was originally archived in [6], and was produced independently of the work in [11], which the authors have only recently managed to obtain a copy. In [11], a geometric argument is provided for the classification of (ordered) oriented fused links via their linking numbers and their virtual linking numbers as defined in [11]; these are different to the virtual linking numbers considered in [7], instead taking a sum of signs over the virtual crossings, with sign of virtual crossings being determined by their order of appearance, and is dependent upon the choice of ordering of the components. The geometric proof in [11] makes use of specific constructions, such as the use of “fusions of virtual Hopf links”. In contrast, in this paper, we provide an algebraic proof of the classification of (unordered) oriented classical links under fused isotopy, with recourse to standard machinery of Alexander’s Theorem, and the theory of braids. Along the way we provide some additional simple results about braids required for the main construction. In the case of classical links, the virtual linking numbers, in the sense of [11], would be zero and the results obtained about the classification are consistent.

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References